II. In any arithmetical series whose first term is equal to the common difference, the sum of the cubes of any number of terms is equal to the product of the common difference by the square of the sum of the series.

Let a, $a + d$, $a + 2d$, ... $a + (n - 1)d$ be any arithmetical series, and let $S =$ the sum of n terms: then

$$S = \tfrac{1}{2}n[2a + (n - 1)d]; \ldots\ldots\ldots\ldots \quad (1)$$

and denoting the sum of the cubes of the terms by S', we have

$$S' = a^3 + (a + d)^3 + (a + 2d)^3 + \ldots. + [a + (n-1)d]^3.$$

Expanding, and adding similar terms gives

$$S' = na^3 + a^2d[3+6+9+\ldots+3(n-1)] + ad^2[3+12+27+\ldots+3(n-1)^2] + d^3[1+8+27+\ldots+(n-1)^3]. \ldots\ldots\ldots\ldots (2)$$

But $3+6+9+\ldots+3(n-1) = \tfrac{3}{2}(n^2-n)$, $3+12+27+\ldots.+3(n-1)^2 = 3[1^2+2^2+3^2+\ldots.+(n-1)^2] = [1+2+3+\ldots.+(n-1)](2n-1) = \tfrac{1}{2}(n^2-n)(2n-1)$, and $1+8+27+\ldots+(n-1)^3 = 1^3+2^3+3^3+\ldots+(n-1)^3 = [\tfrac{1}{2}(n^2-n)]^2$. Substituting these values in equation (2) gives,

$$S' = na^3 + 3a^2d[\tfrac{1}{2}(n^2-n)] + ad^2[\tfrac{1}{2}(n^2-n)](2n-1) + d^3[\tfrac{1}{2}(n^2-n)^2].$$

Performing the operations indicated this becomes,

$$S' = na^3 + \tfrac{3}{2}n^2a^2d - \tfrac{3}{2}na^2d + n^3ad^2 - \tfrac{3}{2}n^2ad^2 + \tfrac{1}{2}nad^2 + \tfrac{1}{4}n^4d^3 - \tfrac{1}{2}n^3d^3 + \tfrac{1}{4}n^2d^3.$$

Factoring, $S' = \tfrac{1}{2}n(2a + nd - d)(a^2 + nad - ad + \tfrac{1}{2}n^2d^2 - \tfrac{1}{2}nd^2)$,

or $$S' = \tfrac{1}{2}n[2a+(n-1)d](a^2+nad-ad+\tfrac{1}{2}n^2d^2-\tfrac{1}{2}nd^2), \ldots \quad (3)$$

in which the factor $\tfrac{1}{2}n[2a + (n - 1)d] = S$. Therefore the sum of the cubes of the terms is divisible by the sum of the series, as announced in Prop. I.

If we square both members of equation (1) we have

$$S^2 = (\tfrac{1}{2}n)^2(4a^2 + 4nad - 4ad + n^2d - 2nd + d^2),$$

which becomes $S^2 = (\tfrac{1}{2}n)^2(n^2d^2 + 2nd^2 + d^2)$ when $a = d$. Making $a = d$ in equation (3) it becomes $S' = (\tfrac{1}{2}n)^2(n^2d^2 + 2nd^2 + d^2)d = S^2d$; which proves Prop. II.

SOLUTION OF NUMERICAL EQUATIONS OF HIGHER DEGREES WITH SECONDARY (IMAGINARY) ROOTS.

BY PROF. A. ZIELINSKI, C. E., AUGUSTA, GEORGIA.

I. Any algebraic equation of the $2n^{\text{th}}$ degree, and having n pairs of secondary conjugate roots, will have primary (real) coefficients, and its general form will be,

(1) $$x^{2n} + A_1x^{2n-1} + \ldots.. + A_{2n-1}x + A_{2n} = 0.$$

The roots of this equation will have the form,

$$(2)\qquad \begin{cases} x_1 = a_1 + b_1 i, & x_2 = a_1 - b_1 i, \\ x_3 = a_2 + b_2 i, & x_4 = a_2 - b_2 i, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ x_{2n-1} = a_n + b_n i, & x_{2n} = a_n - b_n i; \end{cases}$$

where $i = \sqrt{-1}$.

If in (1) we introduce for x one of its values in (2), for instance, $x_1 = a_1 + b_1 i$, we get
$(a_1 + b_1 i)^{2n} + A_1(a_1 + b_1 i)^{2n-1} + \ldots + A_{2n-1}(a_1 + b_1 i) + A_{2n} = 0$;
and developing,

$$\varphi([A], a_1, b_1) + \psi([A], a_1, b_1)i = 0,$$

where $[A]$ stands for $A_1, A_2, \ldots . A_{2n}$.

This equation cannot reduce to zero, unless (3) $\varphi([A], a_1, b_1) = 0$ and (4) $\psi([A], a_1, b_1) = 0$; that is, unless (3) and (4), both of the $2n$th degree, containing two unknown quantities each, reduce simultaneously to zero, id est, for the same corresponding values of a_1 and b_1.

Proceeding as usual in the solution of simultaneous equations, we get

$$(5)\qquad \varphi([A], a_1, b_1) = Q.\psi([A], a_1, b_1) + X(a_1 \text{ or } b_1),$$

where Q is the quotient obtained by the division of $\varphi(\;\;)$ by $\psi(\;\;)$; and $X(a_1$ or $b_1)$ the ultimate remainder, containing one variable only.

From (3), (4) and (5) we get

$$X(a_1 \text{ or } b_1) = 0;$$

and this equation gives us finally the value or the values of a or b, which, introduced in (3) or (4), will give the corresponding value or values of b or a.

II. Any algebraic equation of $(2n+1)$th degree, having n pairs of secondary conjugate roots, and one single secondary root, will have secondary coefficients, and its general form will be,

$$(1)\quad x^{2n+1} + (A_1 + B_1 i)x^{2n} + (A_2 + B_2 i)x^{2n-1} + \ldots + (A_{2n} + B_{2n} i)x + (A_{2n+1} + B_{2n+1} i) = 0.$$

Its roots will have the form,

$$(2)\qquad \begin{cases} x_1 = a_1 + b_1 i, & x_2 = a_1 - b_1 i, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ x_{2n-1} = a_n + b_n i, & x_{2n} = a_n - b_n i, \\ x_{2n+1} = a_{n+1} \pm b_{n+1} i. \end{cases}$$

Eqn. (1) has the form $\quad M + Ni = 0;$
and reduces to zero evidently only when simultaneously

$$M = x^{2n+1} + A_1 x^{2n} + A_2 x^{2n-1} + \ldots\ldots + A_{2n}x + A_{2n+1} = 0, \;\&$$
$$N = \qquad\qquad B_1 x^{2n} + B_2 x^{2n-1} + \ldots\ldots + B_{2n}x + B_{2n+1} = 0.$$

Now it is plain that (1) reduces to zero for the same values of x which satisfy $N = 0$; therefore (1) and $N = 0$ have $2n$ roots in common; hence the

$(2n+1)$th root of (1) will be a quantity the difference between which and x, multiplied by N, gives (1) as result. Dividing therefore (1) by N, we get the single secondary root of (1). It is evident that, if (1) agrees with the hypothesis, N will be the exact divisor of (1).

Again it is evident, from the general theory, that

$$\pm B_1 = \pm b_{n+1}\,; \text{ and } \pm\left(A_1 - \frac{B_2}{B_1}\right) = \mp b_{n+1}$$

If therefore

$$(3) \qquad x - \left[\pm\left(A_1 - \frac{B_2}{B_1}\right) \pm B_1 i\right] = x - (\mp a_{n+1} \pm b_{n+1} i)$$
$$= x \pm (a_{n+1} \pm b_{n+1} i)$$

is an exact divisor of (1), then $\mp(a_{n+1} \pm b_{n+1} i)$ will be the single secondary root of (1), and dividing (1) by (3) we get $N = 0$. But if there is a remainder, the roots of (1) have not the form (2).

The equation $N = 0$, of the $2n$th degree, gives the remaining $2n$ roots, as explained in I.

Examples. *A.* (1) $X = x^7 + (0 - 5i)x^6 - (6 - 15i)x^5 - (32 + 15i)x^4$
$+ (35 + 205i)x^3 + (236 - 790i)x^2 - (558 - 1190i)x + (468 - 780i) = 0.$

We have $N = -5x^6 + 15x^5 - 15x^4 + 205x^3 - 790x^2 + 1190x - 780 = 0$

or $\quad N = x^6 - 3x^5 + 3x^4 - 41x^3 + 158x^2 - 238x + 156 = 0.$

Here $B_1 = b_{n+1} = 5$; $A_1 - B_2 \div B_1 = -a_{n+1} = +3$;

and $\quad a_{n+1} \pm b_{n+1}\sqrt{-1} = -3 \pm 5\sqrt{-1}.$

Now, $X = 0$ is divisible by $x + (3 - 5\sqrt{\ }-)$ only, and therefore $x_7 = -3 + 3\sqrt{-1}$ is a root of (1) and we have

$$X \div (x - x_7) = N = 0.$$

Resolving $N=0$, we see first if there are any primary integral roots, and get $x_1 = 2$; $x_2 = 3$. We have then,

$N \div (x - x_1)(x - x_2) = x^4 + 2x^3 + 7x^2 - 18x + 26 = 0 = C'$;

and introducing in this equation $x = a + b\sqrt{-1}$, we get,

$C = a^4 + 2a^3 - a^2(6b^2 - 7) - 6a(b^2 + 3) + (b^4 - 7b^2 + 2b)$
$+ [4a^3b + 6a^2b - a(4b^3 - 14b) - 18b - 2b^3]\sqrt{-1} = 0$; whence

(*A*) $\quad b^4 - b^2(6a^2 + 6a + 7) + a^4 + 2a^3 + 7a^2 - 18a + 26 = 0,$

(*B*) $\quad - b^2(2a + 1) + 2a^3 + 3a^2 + 7a - 9 = 0,$

and by combining these two equations, we get finally,

(2) $\quad 16a^6 + 48a^5 + 104a^4 + 128a^3 - 35a^2 - 91a - 170 = 0.$

We find as primary integral roots of (2), $a' = +1$, $a'' = 2$; and introducing these values of a in (*B*), we get the corresponding values of b;

$b' = \pm 1$, $b'' = \pm 3$; and the four roots of $C' = 0$ are,
$x_3 = 1 + \sqrt{-1}$, $x_4 = 1 - \sqrt{-1}$, $x_5 = -2 + 3\sqrt{-1}$, $x_6 = -2 - 3\sqrt{-1}$.
Therefore the seven roots of (1) are as follows:

$$x_1 = 2,\ x_2 = 3,\ x_3 = 1 + \sqrt{-1},\ x_4 = 1 - \sqrt{-1}$$
$$x_5 = -2 + 3\sqrt{-1},\ x_6 = -2 - 3\sqrt{-1},\ x_7 = -3 + 5\sqrt{1};$$

and $-(x_1 + x_2 + x_3 + \ldots x_7) = A_1 + B_1\sqrt{-1} = +(0 - 5\sqrt{-1})$;

$$-(x_1 \cdot x_2 \cdot x_3 \ldots x_7) = A_{2n+1} + B_{2n+1}\sqrt{-1} = 468 - 780\sqrt{-1}.$$

B. $X = x^3 - (7 + 5i)x_2 + (19 + 30i)x - (13 + 65i) = 0.$

Here $N = -5x^2 + 30x - 65 = 0$, or $N = x^2 - 6x + 13 = 0$; and
$B_1 = b_{n+1} = +5$, $A_1 - B_2 \div B_1 = -a_{n+1} = -7 + 6 = -1$,

$$a_{n+1} \pm b_{n+1}\sqrt{-1} = 1 \pm 5\sqrt{-1}.$$

$X = 0$ is divisible by $[x - (1 + 5\sqrt{-1})]$ only, therefore $x^3 = 1 + 5\sqrt{-1}$ is a root of $x = 0$, and we have $X \div (x - x_3) = N = 0$.

Introducing in this equation $x = a + bi$, we get

$$N' = a_1^2 - b_1^2 - 6a_1 + 13 + (2a_1b_1 - 6b_1)\sqrt{-1} = 0;$$ and

$(A)\quad a_1^2 - b_1^2 - 6a_1 + 13 = 0,\qquad (B)\quad 2a_1b_1 - 6b_1 = 0.$

(B) gives $a_1 = 3$, and (A) gives $b_1 = \pm 2$; therefore the roots of $N = 0$ are $x_1 = 3 + 2\sqrt{-1}$, $x_2 = 3 - 2\sqrt{-1}$, and the three roots of $X = 0$ are $x_1 = 3 + 2\sqrt{-1}$, $x_2 = 3 - 2\sqrt{-1}$, $x_3 = 1 + 5\sqrt{-1}$; and

$$-(x_1 + x_2 + x_3) = -(7 + 5\sqrt{-1}) = A_1 + B_1\sqrt{-1},$$
$$-(x_1 \cdot x_2 \cdot x_3) = -(13 + 65\sqrt{-1}) = A_{2n+1} + B_{2n+1}\sqrt{-1}.$$

NOTE BY ARTEMAS MARTIN.—I have discovered that the formula given by me at the top of page 119, No. 7, Vol. I of the ANALYST, holds only when $n = 2$.

Since the equation $\sqrt{a} = \frac{a}{r_m}\left[1 - \left(\frac{R_m}{a}\right)\right]^{\frac{1}{2}}$, where $R_m = a - r_m^2$,

is identical, it should be written, to reduce r_m and R_m to integers,

$$\sqrt{a} = \frac{(10)^m a}{(10)^m r_m}\left[1 - \frac{(10)^{2m}\left(\frac{R_m}{a}\right)}{(10)^{2m}}\right]^{\frac{1}{2}}.$$

The formula for the nth root is

$$\sqrt[n]{a} = \frac{a}{r_m}\left[1 - \left(\frac{S_m}{a}\right)\right]^{\frac{1}{n}},\ \text{where } S_m = a^{n-1} - r_m^n;$$

but it does not appear to be of any practical use except when $n = 2$.